GENERAL CASE OF SPECULAR – DIFFUSE REFLECTION OF RADIATION ON THE BOUNDARIES OF A PLANE LAYER UNDER RADIATIVE – CONDUCTIVE HEAT EXCHANGE CONDITIONS

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Linear integral equations describing the angular distribution of the radiation intensity at the boundaries of a semitransparent plane layer are formulated for arbitrary reflection indices. The simpler models of specular and diffuse reflection investigated earlier are obtained as particular cases of the general construction mentioned.

As is known, radiative–conductive heat exchange occurs in materials where two energy transfer mechanisms coexist, heat conduction and radiation. Under such complex energy exchange conditions, the radiation and temperature fields depend essentially on the character of the reflection of the radiation from the body boundaries. In application to semitransparent media, i.e., to materials partially absorbing as well as partially emitting radiation, only the simplest models of specular [1, 4, 7] and diffuse [2-5] reflection have been investigated in the literature. It is shown below that these particular cases can be obtained on the basis of a more general analysis.

The reflection of radiation at the boundary of a medium is quite complex since each incident beam of radiation is reflected in many directions within some solid angle whose axis coincides with the direction of specular reflection because of the surface roughness, and the magnitude depends on the surface properties and the angle of incidence. Let radiant energy with intensity $I_{0\nu}$ be incident at an angle φ within the solid angle $d\omega'$ on a portion dS of a surface (sketch). If $I_{r\nu}(\psi)$ denotes the intensity in the specular direction for reflected radiation, and $I_\nu(\varphi)$ in any other direction defined by the angle φ, then by introducing the reflection index $f_\nu(\psi, \varphi) = I_\nu(\varphi)/I_{r\nu}(\psi)$ and the reflection coefficient $R_\nu(\psi)$ by using the energy conservation law, we obtain

$$R_\nu(\psi) I_{0\nu} \cos\psi \, d\omega' = \int_\omega I_{r\nu}(\psi) f(\psi, \varphi) \cos\varphi \, d\omega. \tag{1}$$

Here ω is the solid angle within which the reflection radiation is propagated so that $d\omega = 2\pi \sin\varphi d\varphi$. We call the quantity

$$\Omega(\psi) = \int_\omega f(\psi, \varphi) \cos\varphi \, d\omega \tag{2}$$

the equivalent solid angle [8]. Using it, we find

$$I_{r\nu}(\psi) = \frac{R_\nu(\psi) \; I_{0\nu} \cos\psi \, d\omega'}{\Omega(\psi)}. \tag{3}$$

Let radiation at all possible angles ψ be incident on dS. Then

$$2\pi \cos\varphi \, d\omega \int_{\psi=0}^{\pi/2} \frac{R_\nu(\psi) I_{0\nu}(\psi)}{\Omega(\psi)} f(\psi, \varphi) \cos\psi \sin\psi \, d\psi$$

is reflected in the direction φ within $d\omega$. Let us introduce the quantity $\varepsilon_\nu(\varphi)B_\nu$, for the characteristic of

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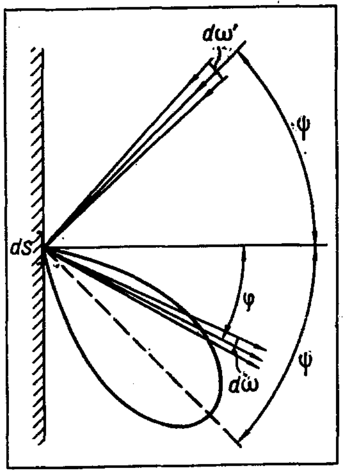

Fig. 1. Character of radiation reflection at the boundary of the layer.

the intrinsic radiation of the boundaries, where B_ν is the spectral surface brightness in the normal direction to the wall, and $\varepsilon_\nu(\varphi)$ characterizes the angular distribution of the energy radiated by the surface. If the layer boundaries are transmitting, then secondary radiation can occur and in that case we consider it to be included in the quantity $\varepsilon_\nu(\varphi)B_\nu$. For simplicity we henceforth omit the subscript ν.

Let us consider the radiation transfer equation

$$\cos\varphi \frac{dI(x,\ \varphi)}{dx} = -kI(x,\ \varphi) + j(x). \tag{4}$$

Letting $I(x,\ \varphi)|_{|\varphi|<\pi/2} = I_+(x,\ \varphi)$, $I(x,\ \varphi+\pi)|_{|\varphi|<\pi/2} = I_-(x,\ \varphi)$, the change in φ in the range $[-\pi/2,\ \pi/2]$ can be examined by solving (4) separately for I_+ and I_-. Since the latter have the sense of radiation intensities going to the coordinate x from the left and right along the selected direction φ, the boundary conditions can be represented as

$$I_+(0,\ \varphi) = \varepsilon_1(\varphi)B_1 + 2\pi \int_0^{\pi/2} \frac{R_1(\psi)}{\Omega_1(\psi)} f_1(\psi,\ \varphi) I_-(0,\ \psi)\sin\psi\cos\psi\, d\psi;$$
$$I_-(h,\ \varphi) = \varepsilon_2(\varphi)B_2 + 2\pi \int_0^{\pi/2} \frac{R_2(\psi)}{\Omega_1(\psi)} f_2(\psi,\ \varphi) I_+(h,\ \psi)\sin\psi\cos\psi\, d\psi. \tag{5}$$

The subscripts 1 and 2 refer to the left and right boundaries, respectively. The solution of (4) appears thus:

$$I_+(x,\ \varphi) = I_+(0,\ \varphi)\, e^{-\frac{kx}{\cos\varphi}} + \int_0^x j(\xi)\, e^{-\frac{k(x-\xi)}{\cos\varphi}} \frac{d\xi}{\cos\varphi}; \tag{6}$$

$$I_-(x,\ \varphi) = I_-(h,\ \varphi)\, e^{-\frac{k(h-x)}{\cos\varphi}} + \int_x^h j(\xi)\, e^{-\frac{k(\xi-x)}{\cos\varphi}} \frac{d\xi}{\cos\varphi}. \tag{7}$$

Solving (5) jointly, and using (6) and (7), we obtain the following integral equations for the quantities $I_+(0,\ \varphi)$ and $I_-(h,\ \varphi)$ which govern the spatial distribution of radiation near the boundaries

$$\begin{aligned} I_+(0,\varphi) = {} & \varepsilon_1(\varphi)B_1 + 2\pi\int_{\psi=0}^{\pi/2} \frac{R_1(\psi)}{\Omega_1(\psi)} f_1(\psi,\ \varphi)\,\varepsilon_2(\psi)B_2 e^{-\frac{kh}{\cos\psi}} \sin\psi\cos\psi\, d\psi \\ & + 2\pi\int_{\psi=0}^{\pi/2}\int_{\xi=0}^{h} \frac{R_1(\psi)}{\Omega_1(\psi)} f_1(\psi,\ \varphi)\, j(\xi)\, e^{-\frac{k\xi}{\cos\psi}} \sin\psi\, d\xi d\psi + 4\pi^2 \int_{\psi=0}^{\pi/2}\int_{\alpha=0}^{\pi/2}\int_{\xi=0}^{h} \frac{R_1(\psi)R_2(\alpha)}{\Omega_1(\psi)\Omega_2(\alpha)} \\ & \times f_1(\psi,\ \varphi) f_2(\alpha,\ \psi)\, j(\xi)\, e^{-\left[\frac{kh}{\cos\psi}+\frac{k(h-\xi)}{\cos\alpha}\right]} \cos\psi - \sin\psi\, \sin\alpha\, d\xi d\alpha d\psi \\ & + 4\pi^2 \int_{\psi=0}^{\pi/2}\int_{\alpha=0}^{\pi/2} I_+(0,\ \alpha) \frac{R_1(\psi)R_2(\alpha)}{\Omega_1(\psi)\Omega_2(\alpha)} f_1(\psi,\ \varphi) f_2(\alpha,\ \psi)\ e^{-\left(\frac{kh}{\cos\psi}+\frac{kh}{\cos\alpha}\right)} \cos\psi\, \sin\psi\, \cos\alpha\, \sin\alpha\, d\alpha\, d\psi; \end{aligned} \tag{8}$$

$$\begin{aligned} I_-(h,\varphi) = {} & \varepsilon_2(\varphi)B_2 + 2\pi\int_{\psi=0}^{\pi/2} \frac{R_2(\psi)}{\Omega_2(\psi)} f_2(\psi,\ \varphi)\,\varepsilon_1(\psi)B_1 e^{-\frac{kh}{\cos\psi}} \cos\psi\sin\psi\, d\psi \\ & + 2\pi\int_{\psi=0}^{\pi/2}\int_{\xi=0}^{h} \frac{R_2(\psi)}{\Omega_2(\psi)} f_2(\psi,\ \varphi)\, j(\xi)\, e^{-\frac{k(h-\xi)}{\cos\psi}} \sin\psi\, d\xi\, d\psi + \\ & + 4\pi^2 \int_{\psi=0}^{\pi/2}\int_{\alpha=0}^{\pi/2}\int_{\xi=0}^{h} \frac{R_2(\psi)R_1(\alpha)}{\Omega_1(\alpha)\Omega_2(\psi)} f_2(\psi,\ \varphi) f_1(\alpha,\ \psi)\, j(\xi)\, e^{-\left(\frac{k\xi}{\cos\alpha}+\frac{kh}{\cos\psi}\right)} \\ & \times \cos\psi\, \sin\psi\, \sin\alpha\, d\xi\, d\alpha\, d\psi + 4\pi^2 \int_{\psi=0}^{\pi/2}\int_{\alpha=0}^{\pi/2} I_-(h,\ \alpha) \frac{R_2(\psi)R_1(\alpha)}{\Omega_1(\alpha)\Omega_2(\psi)} f_1(\psi,\varphi) f_2(\alpha,\psi)\ e^{-\left(\frac{kh}{\cos\alpha}+\frac{kh}{\cos\psi}\right)} \sin\alpha\, \sin\psi\, \cos\alpha\, \cos\psi\, d\alpha\, d\psi. \end{aligned} \tag{9}$$

Here $j(\xi) \equiv j_\nu[T(\xi)]$ is the radiation coefficient, or the volume spectral density of the radiation. This quantity is defined by the equality [9]

$$j_\nu(T) = \frac{k_\nu n_\nu^2}{\pi} I_B(\nu, T). \tag{10}$$

The integral equations (8) and (9) are linear so that if the temperature field of the layer and the reflection characteristics of the boundaries are known, the functions $I_+(0, \varphi)$ and $I_-(h, \varphi)$ can always be found by at least numerical methods. If the temperature distribution in the layer must be found, then (8) and (9) are solved jointly with the fundamental equation describing the temperature field

$$\operatorname{div}(\bar{q} + \bar{E}) + c\gamma \frac{\partial T}{\partial \tau} = 0, \tag{11}$$

where the radiation vector is found from the relationship

$$E(x) = 2\pi \int_{\nu=0}^{\infty} \int_{\varphi=0}^{\pi/2} [I_+(x, \varphi) - I_-(x, \varphi)] \sin\varphi \cos\varphi \, d\varphi \, d\nu. \tag{12}$$

Furthermore, let us examine two extreme cases, diffuse and specular reflection. In the first we should put $f_1(\psi, \varphi) = f_2(\psi, \varphi) = 1$ and should also consider the brightness of the boundaries and the reflection coefficients R_1 and R_2 constant for all directions. If ε_1 and ε_2 are the emissivities of the walls, then $B_1 = \varepsilon_1 n^2 I_B(T_1)$; $B_2 = \varepsilon_2 h^2 I_B(T_2)$, where $I_B(T_1)$, $I_B(T_2)$ are the radiation intensities of a black surface at the temperatures T_1 and T_2. In this case the equivalent solid angles Ω_1 and Ω_2 equal π. Now the right sides of (8) and (9) are independent of φ so that the integral equations degenerate into algebraic. Taking account of the simplifications made we find for $I_+(0)$ and $I_-(h)$ from (8) and (9)

$$I_+(0) = \Big[\varepsilon_1 n^2 I_B(T_1) + 2R_1\varepsilon_2 n^2 I_B(T_2) E_3(kh) + 2R_1 \int_0^h j(\xi) E_2(k\xi)\, d\xi + 4R_1R_2E_3(kh) \int_0^h j(\xi) E_2[k(h-\xi)]\, d\xi\Big] [1 - 4R_1R_2E_3^2(kh)]^{-1}; \tag{13}$$

$$I_-(h) = \Big[\varepsilon_2 n^2 I_B(T_2) + 2R_2\varepsilon_1 n^2 I_B(T_1) E_3(kh) + 2R_2 \int_0^h j(\xi) E_2[k(h-\xi)]\, d\xi + 4R_1R_2E_3(kh) \int_0^h j(\xi) E_2(k\xi)\, d\xi\Big] [1 - 4R_1R_2E_3^2(kh)]^{-1} \tag{14}$$

and, furthermore, by using (12)

$$\begin{aligned} E(x) = 2\pi \int_{\nu=0}^{\infty} \Big([1 - 4R_1R_2E_3^2(k_\nu h)]^{-1} \{ E_3(k_\nu x) [\varepsilon_1 n_\nu^2 I_B(\nu, T_1) \\ + 2R_1\varepsilon_2 n_\nu^2 I_B(\nu, T_2)] - E_3[k_\nu(h-x)] [\varepsilon_2 n_\nu^2 I_B(\nu, T_2) + 2R_1\varepsilon_1 n_\nu^2 I_B(\nu, T_1) E_3(k_\nu h)] + E_3(k_\nu x) \int_{\xi=0}^{h} j(\xi) [2R_1E_2(k_\nu\xi) \\ + 4R_1R_2E_3(k_\nu h) E_3[k_\nu(h-\xi)]\, d\xi - E_3[k_\nu(h-x)] \int_{\xi=0}^{h} j(\xi) [2R_2E_2[k_\nu(h-\xi)] \\ + 4R_1R_2E_3(k_\nu h) E_3(k_\nu\xi)]\, d\xi \} + \int_{\xi=0}^{x} j(\xi) E_2[k_\nu(x-\xi)]\, d\xi - \int_{\xi=x}^{h} j(\xi) E_2(k_\nu(\xi - x))\, d\xi \Big) d\nu \end{aligned} \tag{15}$$

for the radiation vector $\bar{E}(x)$. This expression agrees with the result obtained in [4] for diffuse reflection with the sole difference that the emissivities and reflection coefficients of both boundaries are identical in [4], but the quantities k and n are taken constant over the spectrum. To the accuracy of a transformation

$$\operatorname{div}\bar{E} = \Phi - \Psi, \tag{16}$$

where Φ and Ψ are the radiativity and absorptivity of the substance [10], (15) agrees with the relations found under the assumption of "grayness" of the layer in [2, 3, 5].

Turning to specular reflection, let us note that $f_1(\psi, \varphi) = f_2(\psi, \varphi) = \delta_{\varphi\psi}$, where the subscripts on the symbol δ denote directions within $d\omega$ and $d\omega'$, respectively. In this case

$$\Omega_1 = \Omega_2 = 2\pi \int_{d\omega'} \sin\varphi \cos\varphi \, d\varphi = 2\pi \sin\psi \cos\psi \, d\psi,$$

and the integral equations (8) and (9) again degenerate. The functions $I_+(0, \varphi)$ and $I_-(h, \varphi)$ are easily found

$$I_+(0, \varphi) = \Big[\varepsilon_1(\varphi) B_1 + R_1(\varphi)\varepsilon_2(\varphi) B_2 e^{-\frac{kh}{\cos\varphi}} + \int_0^h R_1(\varphi) j(\xi) e^{-\frac{k\xi}{\cos\varphi}} \frac{d\xi}{\cos\varphi}$$

$$+ \int_0^h R_1(\varphi) R_2(\varphi) j(\xi) e^{-\frac{k(2h-\xi)}{\cos\varphi}} \frac{d\xi}{\cos\varphi}\Big]\Big[1 - R_1(\varphi) R_2(\varphi) e^{-\frac{2kh}{\cos\varphi}}\Big]^{-1}; \tag{17}$$

$$I_-(h, \varphi) = \Big[\varepsilon_2(\varphi) B_2 + R_2(\varphi)\varepsilon_1(\varphi) B_1 e^{-\frac{kh}{\cos\varphi}} + \int_0^h R_2(\varphi) j(\xi) e^{-\frac{k(h-\xi)}{\cos\varphi}} \frac{d\xi}{\cos\varphi}$$

$$+ \int_0^h R_1(\varphi) R_2(\varphi) j(\xi) e^{-\frac{k(h+\xi)}{\cos\varphi}} \frac{d\xi}{\cos\varphi}\Big]\Big[1 - R_1(\varphi) R_2(\varphi) e^{-\frac{2kh}{\cos\varphi}}\Big]^{-1}. \tag{18}$$

Again using (12), we find an expression for the radiation vector which agrees exactly with that obtained in [7]:

$$E(x) = 2\pi \int_{\nu=0}^{\infty} \Big\{ \int_{\xi=0}^{x} j_\nu(\xi) E_2[k_\nu(x-\xi)]\, d\xi - \int_{\xi=x}^{h} j_\nu(\xi) E_2[k_\nu(\xi - x)]\, d\xi \cdot$$

$$+ \int_{\xi=0}^{h}\int_{\varphi=0}^{\pi/2} j_\nu(\xi)\Big[R_1(\varphi) e^{-\frac{k_\nu(x+\xi)}{\cos\varphi}} + R_1(\varphi)R_2(\varphi) e^{-\frac{k_\nu(2h+x-\xi)}{\cos\varphi}}$$

$$- R_2(\varphi) e^{-\frac{k_\nu(2h-x-\xi)}{\cos\varphi}} - R_1(\varphi) R_2(\varphi) e^{-\frac{k_\nu(2h-x+\xi)}{\cos\varphi}}\Big]\Big[1 - R_1(\varphi) R_2(\varphi) e^{-\frac{2k_\nu h}{\cos\varphi}}\Big]^{-1} \sin\varphi \, d\varphi \, d\xi$$

$$+ \int_{\varphi=0}^{\pi/2} \sin\varphi\cos\varphi\Big[1 - R_1(\varphi) R_2(\varphi) e^{-\frac{2k_\nu h}{\cos\varphi}}\Big]^{-1}\Big[\varepsilon_2(\varphi) B_2\Big(R_1(\varphi) e^{-\frac{k_\nu(h+x)}{\cos\varphi}}$$

$$- e^{-\frac{k_\nu(h-x)}{\cos\varphi}}\Big) + \varepsilon_1(\varphi) B_1\Big(e^{-\frac{k_\nu x}{\cos\varphi}} - R_2(\varphi) e^{-\frac{k_\nu(2h-x)}{\cos\varphi}}\Big)\Big] d\varphi \Big\} d\nu. \tag{19}$$

Therefore, the integral equations (8) and (9) obtained include all possible cases of reflection of radiation on the boundaries of a semitransparent medium.

NOTATION

$I_\nu(\varphi)$	is the spectral radiation intensity in a direction governed by the angle φ to the normal;
$R_\nu(\psi)$	is the spectral reflection coefficient in the specular direction;
$f_\nu(\psi, \varphi)$	is the reflection index;
Ω	is the equivalent solid angle;
j_ν	is the volume spectral density of the radiation;
$I_B(\nu, T)$	is the Planck function;
B_ν	is the surface spectral brightness;
$\bar{q} = -\lambda \operatorname{grad} T$	is the heat flux vector;
$E(x)$	is the radiant flux vector (radiation vector);
k_ν	is the spectral absorption coefficient;
n_ν	is the spectral refractive index;
$E_m(x)$	is the integrodifferential function of m-th order;
h	is the layer thickness;
c_γ	is the volume specific heat;
x	is the running space coordinate;
τ	is the time coordinate;
φ, ψ	are the angles of observation;
$\varepsilon_\nu(\varphi)$	is the spectral directed emissivity of surface.

Subscripts

1 and 2 refer to the first and second boundaries, respectively;

r denotes the reflected radiation;

0 is incident radiation.

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